II Semester M.Sc. Degree Examination, June 2015
(CBCS)
MATHEMATICS
M 202 T : Complex Analysis

Time : 3 Hours   Max. Marks : 70

Instructions : *Answer* ***any five full*** *questions.*

1. a) Define analytic function and evaluate $\int_c \frac{e^z}{(z-1)^3} dz$ where $C : |z| = 2$.

   b) Define Mobius transformation. Prove that every Mobius transformation maps circles and straight lines in the z-plane into circles or lines.

   c) If f(z) is continuous in an open set G in the complex plane and $\int_c f(z)dz = 0$ for every simple closed curve in G, then prove that, the function f(z) is analytic on G. Also evaluate $\int_c \frac{e^{2z}}{(z-\pi i)^6} dz$, where C is the boundary of a square whose sides lie along the lines $x = \pm 4$, $y = \pm 4$ described in the positive direction. **(3+4+7)**

2. a) State and prove Cauchy's theorem for a rectangle.

   b) State and prove Liouville's theorem. Deduce the fundamental theorem of algebra. **(6+8)**

3. a) Find the radius of convergence of

   i) $\sum_{n=0}^{\infty} (\log n)^n z^n$ ii) $\sum_{n=1}^{\infty} \frac{n\sqrt{2}+i}{1+2in} z^n$.

   b) Prove that the power series and its derivative have the same radius of convergence.

   c) Find the Laurent's series expansion of $f(z) = \frac{1}{z^2(z-i)}$ in

   i) $0 < |z| < 1$

   ii) $0 < |z - i| < 1$

   iii) $|z - i| > 1$. **(4+6+4)**

4. a) State and prove Taylor's theorem.

   b) Define the terms :

      i) Pole

      ii) Removable singularity

      iii) Essential singularity.

      iv) Isolated singularity and give example for each.

   c) Prove that an analytic function comes arbitrarily close to any complex number in the neighbourhood of an essential singularity. **(5+4+5)**

5. a) If n is a positive integer then show that $\int_0^{2\pi} \cos(n\theta - \sin\theta) e^{\cos\theta}\, d\theta = \frac{2\pi}{n!}$ and $\int_0^{2\pi} e^{\cos\theta} \sin(n\theta - \sin\theta) d\theta = 0$. **4**

   b) Solve any two of the integrals :

      i) $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} dx$

      ii) $\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^{ax}} dx,\ 0 < a < \pi$

      iii) $\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)}$. **(5+5)**

6. a) Outline the argument principle, and explain why it is called by that name.

   b) State and prove open mapping theorem.

   c) State and prove Schwartz's Lemma. **(4+5+5)**

7. a) State and prove Hadamard three circle theorem.

   b) State and prove Phragmen-Lidelöf theorem. **(7+7)**

8. a) Let f(z) be analytic in the region $|z| < \rho$, and let $z = re^{i\theta}$ be any point of this region. Then prove that $f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\theta}) d\phi}{R^2 - 2Rr \cos(\theta + \phi) + r^2}$.

   b) Derive the Jensen's formula with standard notations. **(8+6)**